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SHORTER COMMUNICATION

FULLY DEVELOPED LAMINAR FREE CONVECTION BETWEEN VERTICAL PLATES HEATED ASYMMETRICALLY

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NOMENCLATURE

b, channel width;
c, specific heat;
g, gravitational acceleration;
k, thermal conductivity;
l, channel height;
p, fluid pressure;
p_0, hydrostatic pressure;
q, heat rate per unit area;
T, temperature;
u, axial component of velocity;
x, y, coordinate system, see Fig. 1;
β, thermal expansion coefficient;
μ, dynamic viscosity;
ν, kinematic viscosity;
ρ, density.

Subscripts
1, hotter wall;
2, cooler wall;
0, channel entrance;
max, maximum value.

INTRODUCTION

PREVIOUS studies which deal with fully developed flow in vertical flat channels [1–5] are mainly concerned with combined free and forced convection [1–4] or pure free convection with symmetric wall heating at constant temperature [5]. The purpose of this note is to present, in closed forms, fully developed flow solutions for laminar free convection in a vertical, parallel-plate channel with asymmetric heating. The channel walls are maintained either at uniform heat fluxes (UHF) or uniform wall temperatures (UWT). To account for asymmetric heating of the walls, the ratio of the wall heat fluxes or the ratio of the wall temperature differences (above the ambient) is allowed to vary from 0 to 1. The geometry of flow is illustrated in Fig. 1. Due to fully developed flow the fluid motion, which is generated by buoyancy effects, has an unchanging axial velocity profile in the entire channel. As indicated in [6], such a flow is approached when the channel length (l) is considerably larger than the channel width (b).

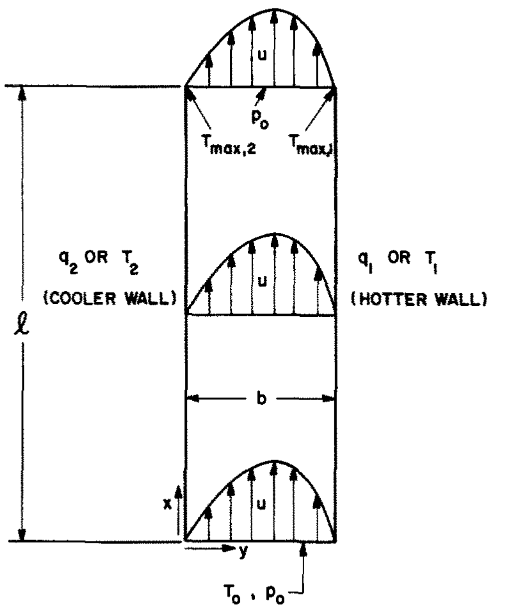

FIG. 1. Flow model.

ANALYSIS AND RESULTS

Under the assumption that $U = U(Y)$ (i.e. fully developed flow), the governing equations for free convection in laminar, two-dimensional, constant property flow [6] are easily reduced to:

$$\frac{d}{dX}P(X) - \frac{d^2}{dY^2}U(Y) = \theta(X, Y) \quad (1)$$

$$Pr\, U(Y)\frac{\partial}{\partial X}\theta(X, Y) = \frac{\partial^2}{\partial Y^2}\theta(X, Y) \quad (2)$$

where

$$U = \frac{b^2 u}{l\nu Gr}, X = \frac{x}{lGr}, Y = \frac{y}{b}$$

$$P = \frac{(p - p_0)b^4}{\rho l^2 \nu^2 Gr^2}, Pr = \frac{\mu c}{k},$$

for UHF,

$$\theta = \frac{T - T_0}{q_1 b/k}, Gr = \frac{g\beta q_1 b^5}{l\nu^2 k}: \quad \text{(UHF)}$$

for UWT,

$$\theta = \frac{T - T_0}{T_1 - T_0}, Gr = \frac{g\beta(T_1 - T_0)b^4}{l\nu^2}. \quad \text{(UWT)}$$

In the above, subscript "1" refers to the hotter wall. Using subscript "2" to denote the cooler wall, the following heat flux and wall temperature difference ratios may be defined:

$$r_H = \frac{q_2}{q_1}, r_T = \frac{T_2 - T_0}{T_1 - T_0}$$

where both r_H and r_T may vary from 0 to 1 (symmetric heating).

Combining equation (1) and equation (2), we have

$$\frac{d^4}{dY^4}U(Y) + Pr\, U(Y)\frac{d^2}{dX^2}P(X) = 0. \quad (3)$$

In equation (3), a solution $U = U(Y)$ is possible only if d^2P/dX^2 is at most a constant, say α. From this equality, by direct integration and applying the conditions $P = 0$ at $X = 0$ and $X = L$ where L is the value of X at $x = l$, that is $L = 1/Gr$, there results

$$P = \frac{\alpha X}{2}(X - L), \quad (4)$$

in which the constant α will be defined later.

Consequently, equation (3) and equation (1) become, respectively,

$$\frac{d^4}{dY^4}U(Y) + \lambda^4 U(Y) = 0 \quad (5)$$

$$\theta(X, Y) = \alpha\left(X - \frac{L}{2}\right) - \frac{d^2}{dY^2}U(Y) \quad (6)$$

where $\lambda = (\alpha\, Pr)^{\frac{1}{4}}$.

(i) *Uniform heat fluxes*

For UHF, the boundary conditions for equation (5) and (6) are

$$U(0) = U(1) = 0 \quad (7a)$$

$$\left(\frac{\partial\theta}{\partial Y}\right)_{Y=0} = -r_H; \quad \left(\frac{\partial\theta}{\partial Y}\right)_{Y=1} = 1. \quad (7b)$$

The general solution of equation (5) is:

$$U(Y) = e^{\lambda Y/s}\left(c_1 \cos\frac{\lambda Y}{s} + c_2 \sin\frac{\lambda Y}{s}\right) + e^{-\lambda Y/s}\left(c_3 \cos\frac{\lambda Y}{s} + c_4 \sin\frac{\lambda Y}{s}\right), \quad (8)$$

where $s = (2)^{\frac{1}{2}}$. Applying conditions (7a), and conditions (7b) in conjunction with equation (6), the constants of integration in equation (8) are obtained:

$$c_1 = -\frac{c_2 e^{\lambda/s} + c_4 e^{-\lambda/s}}{e^{\lambda/s} - e^{-\lambda/s}}\tan\frac{\lambda}{s}, \quad c_2 = \frac{-s - F_3\lambda^3 G}{\lambda^3 H - F_4\lambda^3 G}$$

$$c_3 = -c_1, \quad c_4 = F_3 - c_2 F_4$$

where

$$G = F_2 e^{\lambda/s}\sin\frac{\lambda}{s} + F_2 e^{\lambda/s}\cos\frac{\lambda}{s} + e^{-\lambda/s}\cos\frac{\lambda}{s} + e^{-\lambda/s}\sin\frac{\lambda}{s} - F_2 e^{-\lambda/s}\sin\frac{\lambda}{s} + F_2 e^{-\lambda/s}\cos\frac{\lambda}{s}$$

$$H = F_1 e^{\lambda/s}\sin\frac{\lambda}{s} + F_1 e^{\lambda/s}\cos\frac{\lambda}{s} + e^{\lambda/s}\cos\frac{\lambda}{s} - e^{\lambda/s}\sin\frac{\lambda}{s} - F_1 e^{-\lambda/s}\sin\frac{\lambda}{s} + F_1 e^{-\lambda/s}\cos\frac{\lambda}{s}$$

$$F_1 = \tan\frac{\lambda}{s}\Big/(1 - e^{-\lambda s}), \quad F_2 = \tan\frac{\lambda}{s}\Big/(e^{\lambda s} - 1)$$

$$F_3 = sr_H/[\lambda^3(2F_2 + 1)], \quad F_4 = (2F_1 + 1)/(2F_2 + 1).$$

In order to evaluate the quantity α and hence λ, an energy balance may be performed on an elemental volume of height dx and width b in the channel, with the use of the bulk temperature of the fluid, $T_b = \int_0^b uT dy / \int_0^b u dy$. It then becomes apparent that the bulk temperature of the fluid increases linearly with distance from the channel entrance. (The same result may be obtained by integrating both sides of equation (2) with respect to Y from $Y = 0$ to $Y = 1$.) Since the axial velocity is independent of axial distance, there results $dT_b/dx = \int_0^b u(\partial T/\partial x)\, dy / \int_0^b u dy$. From differentiation of equation (2) with respect to Y from $Y = 0$ to $Y = 1$) Since $\partial T/\partial x = \alpha$. The flow is therefore thermally fully developed. This result follows directly from the single assumption of a fully developed velocity field. The implication is that in free convection in a vertical flat duct, the thermal development length is shorter than or at most equal to that of the hydrodynamic development length, irrespective of the value of the Prandtl number. In pure forced convection, it is of course well known that the relative magnitudes of the thermal and hydrodynamic development distances are strongly influenced by the Prandtl number.

In the present (free convection) case, using the results obtained in the last paragraph and letting $M = \int_0^1 U\,\mathrm{d}Y$, one can show that

$$\frac{\mathrm{d}\theta_b}{\mathrm{d}X} = \alpha = \frac{1 + r_H}{Pr\,M}. \tag{9}$$

To complete the solution, a relation between L and M has to be found. This may be facilitated by evaluating the dimensionless bulk temperature of the fluid at the channel exit:

$$\theta_{b,L} = \frac{\left(\int_0^1 \theta U\,\mathrm{d}Y\right)_{X=L}}{\int_0^1 U\,\mathrm{d}Y}.$$

By equation (6) and from the definition of M,

$$\theta_{b,L} = \frac{\alpha L}{2} - \frac{B(M, r_H)}{M} \tag{10}$$

where

$$B(M, r_H) = \int_0^1 U\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}\,\mathrm{d}Y$$

and can be computed using equation (8). For large values of M (which, as will be seen, correspond to large L), it is found that B is independent of r_H and is related to M by $B = -12(M)^2$. Substituting this relation into equation (10) and with the aid of equation (9) there results: $M = \alpha L/24$. This may be written, by substituting for α from equation (9), as

$$M = 0{\cdot}2887(Pr)^{-\frac{1}{2}}(\bar{L})^{\frac{1}{2}}. \tag{11}$$

In equation (11) and in what follows, a bar over a variable is used to designate its definition using the average of wall heat fluxes (or the average of wall temperature differences for UWT).

Equation (11) indicates that the $\bar{M} \sim \bar{L}$ relation is independent of the wall heat flux ratio. Using equation (11) the temperature distribution in the fully developed flow channel is completely determined when $\bar{L}$ is specified. Thus, with the aid of equation (6) and equation (9), it may be shown that, for large M (or L), the maximum temperatures (occurring at $X = L$) on the two walls are given by

$$\bar{\theta}_{\max,1} = \bar{\theta}_{\max,2} = 6{\cdot}9285(Pr)^{-\frac{1}{2}}(\bar{L})^{\frac{1}{2}}. \tag{12}$$

Using equations (9) and (11), equation (4) may be rephrased as

$$\bar{P} = 2{\cdot}8935(\bar{L})^{\frac{3}{2}}(Pr)^{-1}\left(\frac{\bar{X}}{\bar{L}}\right)\left(\frac{\bar{X}}{\bar{L}} - 1\right) \tag{13a}$$

or

$$\bar{P} = 70{\cdot}1(Pr)^{-1}(M)^3\left(\frac{\bar{X}}{\bar{L}}\right)\left(\frac{\bar{X}}{\bar{L}} - 1\right). \tag{13b}$$

Finally, the following temperature distribution, which is invariant with X, may be evaluated using equation (6):

$$\theta - \theta_1 = \left(\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}\right)_{Y=1} - \left(\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}\right). \tag{14}$$

Note that the present fully developed solution is obtained for large L(or $\bar{L}$). The results in [6] indicate that the present solutions are approached by the developing flow at large L. The limiting values of $\bar{L}$ above which the present results are valid are ascertained in [6]. From the latter, it is found that equation (12) may be used to give a 5 per cent accuracy when $\bar{L} \geqslant 5$.

(ii) *Uniform wall temperatures*

For UWT, it can be deduced from equation (6) that $\alpha = 0$. Thus for UWT in fully developed flows, the pressure is constant. Hence $\lambda = 0$ and equation (5) and equation (6) become, respectively,

$$\frac{\mathrm{d}^4}{\mathrm{d}Y^4}U(Y) = 0 \tag{15}$$

$$\theta = -\frac{\mathrm{d}^2}{\mathrm{d}Y^2}U(Y). \tag{16}$$

The boundary conditions are:

$$U(0) = U(1) = 0, \quad \theta(0) = r_T, \quad \theta(1) = 1.$$

The solutions are:

$$U = (r_T - 1)\frac{Y^3}{6} - r_T\frac{Y^2}{2} + (2r_T + 1)\frac{Y}{6} \tag{17a}$$

$$\theta = (1 - r_T)\,Y + r_T. \tag{17b}$$

The volume flow rate is

$$M = \int_0^1 U\,\mathrm{d}Y = \frac{r_T + 1}{24}. \tag{18}$$

The total heat absorbed by the fluid in traversing the channel is

$$Q'_L = \rho c\left[\int_0^b u(T - T_0)\,\mathrm{d}y\right]_{x=l}$$

which becomes, in dimensionless form,

$$Q_L = \frac{Q'_L b}{\rho c l \nu Gr(T_1 - T_0)} = \left(\int_0^1 U\theta\,\mathrm{d}Y\right)_{X=L} = \frac{4r_T^2 + 7r_T + 4}{180} \tag{19}$$

An average Nusselt number may be defined as

$$Nu = \frac{Q_L' b}{2l(T_1 - T_0) k} = \tfrac{1}{2} Q_L PrGr. \quad (20)$$

Based on the average wall temperature difference $\bar{T} - T_0$ [6], the average Nusselt number becomes

$$\overline{Nu} = \frac{Q_L' b}{2l(\bar{T} - T_0) k} = \tfrac{1}{2} \bar{Q}_L Pr\overline{Gr} = \bar{R}_L Pr\overline{Gr} \quad (21)$$

where

$$\bar{Q}_L = Q_L \left(\frac{2}{1 + r_T}\right)^2, \bar{R}_L = \tfrac{1}{2} \bar{Q}_L = (4r_T^2 + 7r_T + 4) / [90(1 + r_T)^2].$$

The values of $\bar{R}_L$ and $\bar{M}$ for different values of r_T are given in Table 1.

Table 1

r_T	$\bar{R}_L$	$\bar{M}$
1·0	1/24	1/12
0·5	17/405	1/12
0·1	79/1815	1/12
0·0	2/45	1/12

It is seen from Table 1 that $\bar{M}$ is independent of r_T, while $\bar{R}_L$ is weakly dependent on r_T and varies inversely with it. Equation (21), with R_L from Table 1, is approached by the numerical solutions for developing flow at small $Pr\overline{Gr}$ [6] and may be used with confidence when $Pr\overline{Gr} \leq 2$.

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